# A Note on the Abnormality of Realizations of S4LP

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*Justification logics* [Art08] are essentially refined analogs of modal epistemic logics. Whereas a modal epistemic logic uses the formula $\Box F$ to indicate that $F$ is known to be true, a justification logic uses $t{:}F$ instead, where $t$ is a term that describes a 'justification' or proof of $F$. The structure of justification terms $t$ depends on which modal logic needs to be represented in this explicit format. This structure notwithstanding, the formal correspondence between a modal logic ML and its justification counterpart JL is given by a *realization theorem*. It has two directions: first, each theorem of ML can be turned into a theorem of JL by *realizing* all occurrences of the modality $\Box$ with appropriate justification terms; second and vice versa, if all terms in a theorem of JL are replaced with $\Box$, a process called the *forgetful projection*, then the resulting modal formula is provable in ML.

Such correspondences have been established for many normal modal logics between K and S5 (see [Art08]). The first such result was established by Artemov [Art95, Art01] between the modal logic S4 and the so-called Logic of Proofs LP. The idea behind the realization process is that $\Box F$ is interpreted as "there exists a proof of $F$." Under this interpretation, each modal formula becomes a first-order statement with quantifiers over proofs. The realization theorem for a particular justification logic then states that the logic's operations on terms are rich enough to represent all Skolem functions necessary for Skolemizing valid modal statements. It is, therefore, natural to add a restriction that different negative occurrences of $\Box$, which are interpreted as universal quantifiers over proofs, be realized by distinct justification variables since the Skolemization process replaces such quantifiers by distinct Skolem variables. This additional property of realization is called *normality*, and all justification logics that enjoy a realization theorem do enjoy it in the strong sense that every modal theorem can be realized normally.

In a series of papers culminating in [AN05], Artemov and Nogina developed a logic S4LP that combines modal representation of knowledge as in S4 with justification terms of LP. The connection between implicit modal knowability and explicit evidence terms in this logic is given by the *connection principle* $t : F \to \Box F$ that essentially states that "whatever is known for a reason $t$ must be known."

In their very first paper on the subject, Artemov and Nogina posed the following question about the realization theorem for S4LP: *Whether* [S4LP] *enjoys the realization property: given a derivation $D$ in* [S4LP] [...] *one could find a realization $r$ of all occurrences of $\Box$ in $D$* [...] *such that the resulting formula $F^r$ is derivable in* [LP]*?*[1] (see [AN04, Problem 2][2]).

However, there are reasons to doubt whether this formulation is the right one. Suppose, we want to realize a theorem $t{:}\Box F \to s{:}\Box G$. The formulation above suggests that the realization must be of the form $t{:}t'{:}F^r \to s{:}s'{:}G^r$ for some terms $t'$ and $s'$ (or $t{:}x{:}F^r \to s{:}s'{:}G^r$ for some justification variable $x$ and some term $s'$ if the normality condition is imposed). This, however, changes the meaning of terms $t$ and $s$: according to the connection principle, the statements justified by them become stronger; as a result, the assumption is weakened while the conclusion is simultaneously strengthened. In this note, we formalize this objection by proving that the realization theorem for S4LP does not hold if the requirement of normality is imposed.

* Supported by Swiss National Science Foundation grant 200021–117699.

[1] The original problem is more precise in that it is formulated for particular *constant specifications*. However, the phenomenon we are going to describe is completely independent of constant specifications, hence, we omit them from both the formulation of the problem and from the following discussion. In fact, our result holds for an arbitrary constant specification.

[2] The logic was called LPS4 there.

**Theorem 1.** *Theorem* $z{:}\neg\Box(P\vee Q)\to z{:}\neg\Box(P\vee Q)\wedge\neg\Box P$ *of* S4LP*, where* $z$ *is a justification variable,* $P$ *and* $Q$ *are propositional letters, does not have a normal realization in* LP*.*

This example of a theorem without a normal realization is inspired by work of Ghari [Gha09]. The proof of the theorem is based on the semantics developed by Mkrtychev [Mkr97] for LP.

**Definition 2.** *An* M-model $\mathcal{M}$ *is a pair* $(\mathcal{E},V)$*, where* $V\colon \mathrm{Prop}\to\{\mathit{True},\mathit{False}\}$ *is a* valuation function *and* $\mathcal{E}\colon \mathrm{Tm}\to 2^{\mathrm{Fm}}$ *is an* evidence function *that satisfies several closure conditions that can be found in [Mkr97] and are omitted here for space considerations. Truth for propositional letters and for Boolean connectives is defined in the standard way;* $\mathcal{M}\Vdash t:F$ *iff* $\mathcal{M}\Vdash F$ *and* $F\in\mathcal{E}(t)$*.*

**Theorem 3 ([Mkr97]).** *A justification formula is a theorem of* LP *iff it is valid in all M-models.*

Instead of giving the full definition of closure conditions, we will use the following lemma that easily follows from them.

**Lemma 4.** *For any requirements* $F_i\in\mathcal{E}(t_i)$*,* $i=1,\dots,n$*, on the evidence function, there exists a unique minimal function that satisfies the requirements. Moreover, for this minimal function* $\mathcal{E}(x)=\{F_i\,|\,x=t_i\}$ *for any justification variable* $x$*.*

*Proof (of Theorem 1.).* The normality condition requires both negative occurrences of $\Box$ in the given theorem of S4LP to be realized by distinct justification variables, say $x$ and $y$, whereas the only positive $\Box$ can be realized by an arbitrary term $t$:

$$z{:}\neg t{:}(P\vee Q)\to z{:}\neg x{:}(P\vee Q)\wedge\neg y{:}P\ . \tag{1}$$

It is easy to refute (1) if $t\neq x$. Indeed, let $V(P)=V(Q)=\mathit{False}$ and let $\mathcal{E}$ be the minimal evidence function such that $\neg t{:}(P\vee Q)\in\mathcal{E}(z)$. Then, $\mathcal{M}\nVdash t{:}(P\vee Q)$ simply because both $P$ and $Q$ are false. Therefore, the antecedent of the implication holds. However, for $t\neq x$, the first conjunct in the consequent is false since, by Lemma 4, $\neg x{:}(P\vee Q)\notin\mathcal{E}(z)$.

Thus, $t=x$, and the only normal realization possible is $z{:}\neg x{:}(P\vee Q)\to z{:}\neg x{:}(P\vee Q)\wedge\neg y{:}P$. Here is a model that refutes it. Let $V(P)=\mathit{True}$ and $\mathcal{E}$ be the minimal evidence function such that $P\in\mathcal{E}(y)$ and $\neg x{:}(P\vee Q)\in\mathcal{E}(z)$. Then, by Lemma 4, $P\vee Q\notin\mathcal{E}(x)$, hence, $\mathcal{M}\nVdash x{:}(P\vee Q)$, which is sufficient to make the antecedent true. On the other hand, the second conjunct of the consequent is clearly false. ☐

Whether the realization theorem holds for S4LP without the normality condition remains an open question.

## References


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