

Corrections to “Justification Counterpart of Distributed Knowledge Systems”

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This manuscript contains corrections to the author’s paper titled “Justification Counterpart of Distributed Knowledge Systems” [1]. There are two errors in the paper: in the statement of Lemma 1 and in the proof of Theorem 4.

Distributed Justification Logics

There is a typo in the statement of Lifting Lemma (Lemma 1 , page 30, [1]). It should be read as follows:

Lemma 1 (Lifting Lemma). *For each $*$ $\in \{1, \dots, n, \mathcal{D}\}$, the following statements are provable:*

1. *If $\llbracket t_1 \rrbracket_* A_1, \dots, \llbracket t_m \rrbracket_* A_m, B_1, \dots, B_l \vdash F$ in $\mathbf{JS4}_n^{\mathcal{D}}(\mathcal{CS})$, then*

$$\llbracket t_1 \rrbracket_* A_1, \dots, \llbracket t_m \rrbracket_* A_m, \llbracket x_1^* \rrbracket_* B_1, \dots, \llbracket x_l^* \rrbracket_* B_l \vdash \llbracket p(\vec{t}, \vec{x}) \rrbracket_* F \quad (\dagger)$$

in $\mathbf{JS4}_n^{\mathcal{D}}(\mathcal{CS}')$, for some justification variables x_i^ (in Var^*), term $p(\vec{t}, \vec{x})$ in Tm_* and $\mathcal{CS}' \supseteq \mathcal{CS}$ (all $*$ ’s in (\dagger) stand for the same agent).*

2. *In part (1), if \mathcal{CS} is axiomatically appropriate, then (\dagger) is provable in $\mathbf{JS4}_n^{\mathcal{D}}(\mathcal{CS})$.*

We give a proof of the Internalization Lemma below.

Lemma 2 (Internalization Lemma). *Let \mathcal{CS} be an axiomatically appropriate constant specification. If $\mathbf{JL}^{\mathcal{D}}(\mathcal{CS}) \vdash F$, then for each $*$ $\in \{1, \dots, n, \mathcal{D}\}$ there is a term p in Tm_* such that $\mathbf{JL}^{\mathcal{D}}(\mathcal{CS}) \vdash \llbracket p \rrbracket_* F$.*

Proof. By induction on the derivation of F . If F is an axiom, then, since \mathcal{CS} is axiomatically appropriate, there is a constant $c^i \in Tm_i$ such that $\llbracket c^i \rrbracket_i F \in \mathcal{CS}$, for each $i \in G$. Hence, $\llbracket c^i \rrbracket_i F$ is derivable in $\mathbf{JL}^{\mathcal{D}}(\mathcal{CS})$, for each $i \in G$. Moreover, using axiom $\llbracket c^i \rrbracket_i F \rightarrow \llbracket c^i \rrbracket_{\mathcal{D}} F$, we can derive $\llbracket c^i \rrbracket_{\mathcal{D}} F$. If F is obtained by Modus Ponens from G and $G \rightarrow F$, then by the induction hypothesis there are terms $t, s \in Tm_*$ such that $\llbracket t \rrbracket_* G$ and $\llbracket s \rrbracket_*(G \rightarrow F)$ are provable. By axiom **A2**, we derive $\llbracket s \cdot_* t \rrbracket_* F$. If $F = \llbracket c_{j_m}^{i_m} \rrbracket_{i_m} \dots \llbracket c_{j_1}^{i_1} \rrbracket_{i_1} A \in \mathcal{CS}$, is obtained by the Iterated Axiom Necessitation Rule, then, since \mathcal{CS} is upward closed, there is a constant $c^i \in Tm_i$ such that $\llbracket c^i \rrbracket_i \llbracket c_{j_m}^{i_m} \rrbracket_{i_m} \dots \llbracket c_{j_1}^{i_1} \rrbracket_{i_1} A$ is in \mathcal{CS} , and therefore is derivable in $\mathbf{JL}^{\mathcal{D}}(\mathcal{CS})$. Moreover, using axiom **A3**, we can derive $\llbracket c^i \rrbracket_{\mathcal{D}} \llbracket c_{j_m}^{i_m} \rrbracket_{i_m} \dots \llbracket c_{j_1}^{i_1} \rrbracket_{i_1} A$.

Now, suppose the logics $\mathbf{JS4}_n^{\mathcal{D}}$ and $\mathbf{JS5}_n^{\mathcal{D}}$ are formulated by the rule **R3**, instead of **R2**. If $F = \llbracket c^i \rrbracket_i A \in \mathcal{CS}$, is obtained by the Axiom Necessitation Rule, then use axiom **A6** to derive $\llbracket !_i c^i \rrbracket_i A$ in $\mathbf{JL}^{\mathcal{D}}(\mathcal{CS})$. Moreover, using axiom **A3**, we can derive $\llbracket !_i c^i \rrbracket_{\mathcal{D}} A$.

Finally note that, as you can see from the above proof, the term p depends on the agent $*$. □

Realization Theorem

Remo Goetschi (personal communication) pointed out that the proof of the Realization Theorem (Theorem 4, page 34, [1]) fails. Here, I briefly express the reason. Consider the rule ($R\Box$):

$$\frac{wRv, \Gamma \Rightarrow \Delta, v \Vdash A}{\Gamma \Rightarrow \Delta, w \Vdash \Box A} (R\Box)$$

By the induction hypothesis, we know that the premise is valid in every strong model \mathcal{M} . Then, by the Fully Explanatory Property, we find a term t such that the formula $\llbracket t \rrbracket A$ holds in world $[w]$ of \mathcal{M} . Note that this term t depends on $[w]$. Therefore, in general, $\llbracket t \rrbracket A$ does not hold in other models or even in other worlds of \mathcal{M} . Hence t is not a suitable term to realize the displayed essential \Box in the conclusion of ($R\Box$).

In the following, we fix the realization theorem for \mathbf{T}_n^D . We first construct a finite canonical pseudo-Fitting model for justification logic \mathbf{JT}_n^D which enjoys the fully explanatory property. Then, using Hakli-Negri's labelled sequent system $\mathbf{G3T}_n^D$, we present a proof for the realization theorem $\mathbf{JT}_n^{D^\circ} = \mathbf{T}_n^D$. We prefer \mathbf{JT}_n^D because this simplifies the proof of completeness with respect to fully explanatory finite models.

Given a formula F of \mathbf{JT}_n^D , let $Sub(F)$ denote the set of all subformulas of F . We define

$$\begin{aligned} Sub(\llbracket t \rrbracket_i F) &= \{\llbracket t \rrbracket_i F, \llbracket t \rrbracket_{\mathcal{D}} F\} \cup Sub(F) \\ Sub(\llbracket t \rrbracket_{\mathcal{D}} F) &= \{\llbracket t \rrbracket_{\mathcal{D}} F\} \cup Sub(F) \end{aligned} \quad (1)$$

Then put

$$X_F = Sub(F) \cup \{\neg G \mid G \in Sub(F)\}.$$

Assume that \mathcal{CS} is a fixed constant specification for \mathbf{JT}_n^D .

Definition 1. A set of formulas S is called *F-maximal $\mathbf{JT}_n^D(\mathcal{CS})$ -consistent* iff

1. $S \subseteq X_F$,
2. For all $G \in Sub(F)$, either $G \in S$ or $\neg G \in S$,
3. whenever $S = \{A_1, \dots, A_n\}$ then $\mathbf{JT}_n^D(\mathcal{CS}) \not\vdash \neg(A_1 \wedge \dots \wedge A_n)$.

It should be noted that, since X_F is a finite set, the number of F-maximal $\mathbf{JT}_n^D(\mathcal{CS})$ -consistent sets are finite. Now, let

$$Con_{\mathbf{JT}_n^D(\mathcal{CS})}(S) = \{A \mid \mathbf{JT}_n^D(\mathcal{CS}), S \vdash A\}$$

be consequences of S in $\mathbf{JT}_n^D(\mathcal{CS})$. we have

Lemma 3. Suppose $S \subseteq X_F$ is an F-maximal $\mathbf{JT}_n^D(\mathcal{CS})$ -consistent, and $\Gamma = Con_{\mathbf{JT}_n^D(\mathcal{CS})}(S)$. Then

1. For all $G \in Sub(F)$, exactly one of G or $\neg G$ is in S ,
2. If $\Lambda \subseteq X_F$ is $\mathbf{JT}_n^D(\mathcal{CS})$ -consistent, then there is an F-maximal $\mathbf{JT}_n^D(\mathcal{CS})$ -consistent set S such that $\Lambda \subseteq S$,
3. If $A \in Sub(F)$ and $A \in \Gamma$, then $A \in S$,
4. If $\mathbf{JT}_n^D(\mathcal{CS}), \Gamma \vdash A$ then $\mathbf{JT}_n^D(\mathcal{CS}), S \vdash A$ and $A \in \Gamma$,
5. Γ is F-maximal $\mathbf{JT}_n^D(\mathcal{CS})$ -consistent,
6. If $A, A \rightarrow B \in \Gamma$ then $B \in \Gamma$.

Proof. The proof of Items 1, 2 and 3 are standard. We prove other items.

4. Suppose $\mathbf{JT}_n^D(\mathcal{CS}), \Gamma \vdash A$, and consider a proof $\sigma_1, \dots, \sigma_n$ of A , where $\sigma_n = A$. Note that if σ_i is in Γ then, by the definition of Γ , we have $\mathbf{JT}_n^D(\mathcal{CS}), S \vdash \sigma_i$. Next by replacing

each assumption $\sigma_i \in \Gamma$ by its proof in $\mathbf{JT}_n^D(\mathcal{CS}), S$, we get a proof of A in $\mathbf{JT}_n^D(\mathcal{CS}), S$. Thus $\mathbf{JT}_n^D(\mathcal{CS}), S \vdash A$ and $A \in \Gamma$.

5. Since S is F-maximal, Γ is too. Now suppose that Γ is not consistent, and hence $\mathbf{JT}_n^D(\mathcal{CS}), \Gamma \vdash \perp$. By Item 1, we have $\mathbf{JT}_n^D(\mathcal{CS}), S \vdash \perp$, which contradicts the consistency of S .

6. Suppose $A, A \rightarrow B \in \Gamma$. Thus $\mathbf{JT}_n^D(\mathcal{CS}), S \vdash A, A \rightarrow B$, and therefore $\mathbf{JT}_n^D(\mathcal{CS}), S \vdash B$, which implies that $B \in \Gamma$. \square

We call a \mathbf{JT}_n^D -model \mathcal{M} *finite*, if the set of possible worlds of \mathcal{M} is finite.

Theorem 1. *For a given constant specification \mathcal{CS} , the justification logic \mathbf{JT}_n^D is sound and complete with respect to its finite pseudo-Fitting models that respecting \mathcal{CS} .*

Proof. Soundness follows from Completeness Theorem (Theorem 2, page 31, [1]). For completeness, suppose F is a \mathbf{JT}_n^D -formula and $\mathbf{JT}_n^D(\mathcal{CS}) \not\vdash F$. We construct a finite canonical model $\mathcal{M}_F = (\mathcal{W}, \mathcal{R}_*, \mathcal{E}_*, \Vdash_p)$ for $\mathbf{JT}_n^D(\mathcal{CS})$ such that $\mathcal{M}_F \not\vdash_p F$. Define \mathcal{W} as follows:

$$\mathcal{W} = \{\Gamma \mid \Gamma = \text{Con}_{\mathbf{JT}_n^D(\mathcal{CS})}(S), \text{ for some F-maximal } \mathbf{JT}_n^D(\mathcal{CS})\text{-consistent set } S\}$$

Notice that \mathcal{W} is finite. Definitions of \mathcal{R}_* , \mathcal{E}_* and \Vdash_p are as follows:

$$\begin{aligned} \Gamma \mathcal{R}_* \Delta &\iff S^{\natural*} \subseteq T, \\ \Gamma \in \mathcal{E}_*(t, A) &\iff \llbracket t \rrbracket_* A \in \Gamma, \\ (\mathcal{M}_F, \Gamma) \Vdash_p P &\iff P \in S \end{aligned} \tag{2}$$

where $\Gamma = \text{Con}_{\mathbf{JT}_n^D(\mathcal{CS})}(S)$, $\Delta = \text{Con}_{\mathbf{JT}_n^D(\mathcal{CS})}(T)$, P is a propositional letter and $S^{\natural*} = \{A \mid \llbracket t \rrbracket_* A \in S, \text{ for some term } t \in \text{Tm}_*\}$. Forcing relation \Vdash_p on arbitrary formulas is defined as in Definition 5 in [1]. The Truth Lemma can be shown for subformulas of F , i.e., for each formula $A \in \text{Sub}(F)$ and each $\Gamma \in \mathcal{W}$ such that $\Gamma = \text{Con}_{\mathbf{JT}_n^D(\mathcal{CS})}(S)$, we have

$$(\mathcal{M}_F, \Gamma) \Vdash_p A \iff A \in S.$$

It is easy to see that \mathcal{M} is a model of \mathbf{JT}_n^D , i.e., each \mathcal{R}_* is reflexive and \mathcal{E}_* satisfies $\mathcal{E}1$, $\mathcal{E}2$ and $\mathcal{E}3$ (and $\mathcal{E}4$, if $n = 1$), and \mathcal{M}_F respects \mathcal{CS} .

Finally, since $\mathbf{JT}_n^D(\mathcal{CS}) \not\vdash F$, the set $\{\neg F\} \subseteq X_F$ is $\mathbf{JT}_n^D(\mathcal{CS})$ -consistent. Thus, it can be extended to an F-maximal $\mathbf{JT}_n^D(\mathcal{CS})$ -consistent set S . Let $\Gamma = \text{Con}_{\mathbf{JT}_n^D(\mathcal{CS})}(S)$. Hence $\Gamma \in \mathcal{W}$ and $F \notin S$. Then, by the Truth Lemma, $(\mathcal{M}_F, \Gamma) \not\vdash_p F$, therefore $\mathcal{M}_F \not\vdash_p F$. Indeed, we have shown that if $(\mathcal{M}_F, \Gamma) \Vdash F$ then $\mathbf{JT}_n^D(\mathcal{CS}) \vdash F$. \square

Note that in the canonical model \mathcal{M}_F we have $\mathcal{R}_D \subseteq \cap_{i=0}^n \mathcal{R}_i$, and for $n = 1$, $\mathcal{R}_D = \mathcal{R}_1$. The proof is as follows. Suppose $\Gamma \mathcal{R}_D \Delta$ and $\llbracket t \rrbracket_i A \in S$, for some $i \in G$, where $\Gamma = \text{Con}_{\mathbf{JT}_n^D(\mathcal{CS})}(S)$ and $\Delta = \text{Con}_{\mathbf{JT}_n^D(\mathcal{CS})}(T)$. It suffices to show that $A \in T$. Since $\llbracket t \rrbracket_i A \in \text{Sub}(F)$, by (1) we have $\llbracket t \rrbracket_D A \in \text{Sub}(F)$. Thus, $\llbracket t \rrbracket_D A \in S$. Indeed, otherwise $\neg \llbracket t \rrbracket_D A \in S$, and hence $\neg \llbracket t \rrbracket_D A \in \Gamma$. On the other hand, $\llbracket t \rrbracket_i A \in S$ yields that $\llbracket t \rrbracket_D A \in \Gamma$, that is a contradiction. Now, $\Gamma \mathcal{R}_D \Delta$ and $\llbracket t \rrbracket_D A \in S$ imply that $A \in \Gamma$. Therefore, $\Gamma \mathcal{R}_i \Delta$.

Now we show that the finite canonical model \mathcal{M}_F satisfies a restricted form of the fully explanatory property.

Proposition 1. *Suppose \mathcal{CS} is an axiomatically appropriate constant specification, F is a \mathbf{JT}_n^D -formula and $\mathcal{M}_F = (\mathcal{W}, \mathcal{R}_*, \mathcal{E}_*, \Vdash_p)$ is a finite canonical model. For each $\Gamma \in \mathcal{W}$ and each $A \in \text{Sub}(F)$,*

1. *if for every $\Delta \in \mathcal{W}$ such that $\Gamma \mathcal{R}_* \Delta$ we have $(\mathcal{M}, \Delta) \Vdash_p A$, then for some term $t \in \text{Tm}_*$ we have $(\mathcal{M}, \Gamma) \Vdash_p \llbracket t \rrbracket_* A$,*

2. if for every $\Delta \in \mathcal{W}$ such that $\Gamma\mathcal{R}_1\Delta, \dots, \Gamma\mathcal{R}_n\Delta$ we have $(\mathcal{M}, \Delta) \Vdash_p A$, then for some term $t \in Tm_{\mathcal{D}}$ we have $(\mathcal{M}, \Gamma) \Vdash_p \llbracket t \rrbracket_{\mathcal{D}}A$.

Proof. Assume that $\Gamma = Con_{\mathbf{JT}_n^{\mathcal{D}}(\mathcal{CS})}(S)$ and $A \in Sub(F)$.

(1) Suppose $*$ $\in \{1, \dots, n, \mathcal{D}\}$ and $(\mathcal{M}_F, \Delta) \Vdash_p A$ for every Δ such that $\Gamma\mathcal{R}_*\Delta$. Suppose towards a contradiction that there is no justification term $t \in Tm_*$ such that $(\mathcal{M}_F, \Gamma) \Vdash_p \llbracket t \rrbracket_*A$. Then, the set $S^{\natural*} \cup \{-A\}$ would have to be $\mathbf{JT}_n^{\mathcal{D}}(\mathcal{CS})$ -consistent. Indeed, otherwise

$$\mathbf{JT}_n^{\mathcal{D}}(\mathcal{CS}) \vdash G_1 \rightarrow (G_2 \rightarrow \dots \rightarrow (G_n \rightarrow A) \dots),$$

for some $\llbracket t_1 \rrbracket_*G_1, \dots, \llbracket t_n \rrbracket_*G_n \in S$. By the Internalization Lemma 2, we would obtain a term s such that

$$\mathbf{JT}_n^{\mathcal{D}}(\mathcal{CS}) \vdash \llbracket s \rrbracket_*(G_1 \rightarrow (G_2 \rightarrow \dots \rightarrow (G_n \rightarrow A) \dots)).$$

By axiom **A2**,

$$\mathbf{JT}_n^{\mathcal{D}}(\mathcal{CS}) \vdash \llbracket t_1 \rrbracket_*G_1 \rightarrow (\llbracket t_2 \rrbracket_*G_2 \rightarrow \dots \rightarrow (\llbracket t_n \rrbracket_*G_n \rightarrow \llbracket s \cdot_* t_1 \cdot_* \dots \cdot_* t_n \rrbracket_*A) \dots).$$

Hence $\mathbf{JT}_n^{\mathcal{D}}(\mathcal{CS}), S \vdash \llbracket t \rrbracket_*A$, for $t = s \cdot_* t_1 \cdot_* \dots \cdot_* t_n$, and therefore $\llbracket t \rrbracket_*A \in \Gamma$. Thus, by (2), we have $\Gamma \in \mathcal{E}_*(t, A)$. Hence, $(\mathcal{M}_F, \Gamma) \Vdash_p \llbracket t \rrbracket_*A$, a contradiction. Now since $S^{\natural*} \cup \{-A\}$ is a subset of X_F and is $\mathbf{JT}_n^{\mathcal{D}}(\mathcal{CS})$ -consistent, it could be extended to an F-maximal $\mathbf{JT}_n^{\mathcal{D}}(\mathcal{CS})$ -consistent set T . Let $\Delta = Con_{\mathbf{JT}_n^{\mathcal{D}}(\mathcal{CS})}(T)$. Since $S^{\natural*} \subseteq T$, by (2), we have $\Gamma\mathcal{R}_*\Delta$. But since $A \notin T$, by the Truth Lemma, $(\mathcal{M}_F, \Delta) \not\Vdash_p A$, which contradicts the assumption.

(2) The proof is similar to (1). Suppose $(\mathcal{M}, \Gamma) \not\Vdash_p \llbracket t \rrbracket_{\mathcal{D}}A$, for each $t \in Tm_{\mathcal{D}}$, and for every $\Delta \in \mathcal{W}$ such that $\Gamma\mathcal{R}_1\Delta, \dots, \Gamma\mathcal{R}_n\Delta$ we have $(\mathcal{M}, \Delta) \Vdash_p A$. We show that there is $\Delta \in \mathcal{W}$ with $\Gamma\mathcal{R}_{\mathcal{D}}\Delta$ such that $(\mathcal{M}, \Delta) \not\Vdash_p A$. Consider the set

$$Y = \{G \mid \llbracket t \rrbracket_{\mathcal{D}}G \in S, \text{ for some } t \in Tm_{\mathcal{D}}\} \cup \{-A\}.$$

We prove that Y is consistent. Otherwise, for some $\llbracket t_1 \rrbracket_{\mathcal{D}}G_1, \dots, \llbracket t_m \rrbracket_{\mathcal{D}}G_m$ in S , we have

$$\mathbf{JT}_n^{\mathcal{D}}(\mathcal{CS}) \vdash G_1 \rightarrow (G_2 \rightarrow \dots \rightarrow (G_m \rightarrow A) \dots).$$

Since the constant specification \mathcal{CS} is axiomatically appropriate, by Lemma 2, there is a term s in $Tm_{\mathcal{D}}$ such that

$$\mathbf{JT}_n^{\mathcal{D}}(\mathcal{CS}) \vdash \llbracket s \rrbracket_{\mathcal{D}}(G_1 \rightarrow (G_2 \rightarrow \dots \rightarrow (G_m \rightarrow A) \dots)).$$

By axiom **A2**,

$$\mathbf{JT}_n^{\mathcal{D}}(\mathcal{CS}) \vdash \llbracket t_1 \rrbracket_{\mathcal{D}}G_1 \rightarrow (\llbracket t_2 \rrbracket_{\mathcal{D}}G_2 \rightarrow \dots \rightarrow (\llbracket t_m \rrbracket_{\mathcal{D}}G_m \rightarrow \llbracket s \rrbracket_{\mathcal{D}}A) \dots),$$

for $t = s \cdot_{\mathcal{D}} t_1 \cdot_{\mathcal{D}} \dots \cdot_{\mathcal{D}} t_m$. Hence $\mathbf{JT}_n^{\mathcal{D}}(\mathcal{CS}), S \vdash \llbracket t \rrbracket_{\mathcal{D}}A$, and therefore $\llbracket t \rrbracket_{\mathcal{D}}A \in \Gamma$. Thus, by (2), we have $\Gamma \in \mathcal{E}_{\mathcal{D}}(t, A)$. On the other hand, for each $\Delta \in \mathcal{W}$ such that $\Gamma\mathcal{R}_{\mathcal{D}}\Delta$, since $\mathcal{R}_{\mathcal{D}} \subseteq \cap_{i=0}^n \mathcal{R}_i$, we have $\Gamma\mathcal{R}_i\Delta$, for all $i \in G$. Thus, by the assumption, $(\mathcal{M}, \Delta) \not\Vdash_p A$. This implies that, $(\mathcal{M}, \Gamma) \not\Vdash_p \llbracket t \rrbracket_{\mathcal{D}}A$, which is a contradiction. Thus Y is a consistent set. Extend Y to a maximal consistent set T , and let $\Delta = Con_{\mathbf{JT}_n^{\mathcal{D}}(\mathcal{CS})}(T)$. Therefore, by the truth lemma $(\mathcal{M}, \Delta) \not\Vdash_p A$. On the other hand, it is obvious that $\Gamma\mathcal{R}_{\mathcal{D}}\Delta$, and since $\mathcal{R}_{\mathcal{D}} \subseteq \cap_{i=0}^n \mathcal{R}_i$, we have $\Gamma\mathcal{R}_i\Delta$, for each $i \in G$. \square

It is worth noting that the proof of the above proposition is not constructive, and moreover, the term t satisfied $(\mathcal{M}_F, \Gamma) \Vdash_p \llbracket t \rrbracket A$, depends on world Γ and formula A .

Now, we present a realization algorithm using labelled systems for distributed knowledge system $\mathbf{T}_n^{\mathcal{D}}$. Before stating the realization theorem we need some definitions.

Definition 2. For a $\mathbf{JL}^{\mathbf{D}}$ -formula F , the forgetful projection of F , denoted by F° , is defined inductively as follows:

1. For propositional letter P , $P^\circ = P$, and $\perp^\circ = \perp$,
2. $(A \rightarrow B)^\circ = A^\circ \rightarrow B^\circ$,
3. $(\llbracket t \rrbracket_i A)^\circ = \mathcal{K}_i A^\circ$,
4. $(\llbracket t \rrbracket_{\mathcal{D}} A)^\circ = \mathcal{D} A^\circ$.

For a set X of justification formulas we let $X^\circ = \{F^\circ \mid F \in X\}$.

One direction of the realization theorem is easy, and can be proved for all distributed knowledge systems $\mathbf{JL}^{\mathbf{D}}$.

Lemma 4. For any formula F of $\mathbf{JL}^{\mathbf{D}}$, if $\mathbf{JL}^{\mathbf{D}} \vdash F$ then $\mathbf{L}^{\mathbf{D}} \vdash F^\circ$.

Proof. By induction on a derivation of F in $\mathbf{JL}^{\mathbf{D}}$. If F is an axiom of $\mathbf{JL}^{\mathbf{D}}$, then it is easy to verify that F° is provable in $\mathbf{L}^{\mathbf{D}}$. For instance, $(\llbracket t \rrbracket_i A \rightarrow \llbracket t \rrbracket_{\mathcal{D}} A)^\circ = \mathcal{K}_i A^\circ \rightarrow \mathcal{D} A^\circ$, and $\mathcal{K}_i A^\circ \rightarrow \mathcal{D} A^\circ$ is an axiom of $\mathbf{L}^{\mathbf{D}}$. If F is obtained by Modus Ponens from G and $G \rightarrow F$, then by the induction hypothesis G° and $G^\circ \rightarrow F^\circ$ are provable in $\mathbf{L}^{\mathbf{D}}$. Thus, F° is provable in $\mathbf{L}^{\mathbf{D}}$. If $F = \llbracket c_{j_m}^{i_m} \rrbracket_{i_m} \dots \llbracket c_{j_1}^{i_1} \rrbracket_{i_1} A \in \mathcal{CS}$, is obtained by the Iterated Axiom Necessitation Rule, then, since A is an axiom of $\mathbf{JL}^{\mathbf{D}}$, A° is provable in $\mathbf{L}^{\mathbf{D}}$, and hence, by iterated applications of the Necessitation Rule, we can derive $\mathcal{K}_{i_m} \dots \mathcal{K}_{i_1} A^\circ$. Likewise, If $F = \llbracket c^i \rrbracket_i A \in \mathcal{CS}$, is obtained by the Axiom Necessitation Rule, then use the Necessitation Rule to derive $\mathcal{K}_i A^\circ$. \square

For the other direction, let us recall the definition of a realization first.

Definition 3. Let A be a formula in the language of $\mathbf{L}^{\mathbf{D}}$. A realization of the formula A is a $\mathbf{JL}^{\mathbf{D}}$ -formula A^r such that $(A^r)^\circ = A$.

More precisely, a realization A^r is obtained by substituting each modality \mathcal{K}_i in A for a term in Tm_{i_1} , and each modality \mathcal{D} in A for a term in $Tm_{\mathcal{D}}$. A realization is called *normal* if all negative occurrences of modalities are replaced by distinct variables.

Definition 4. let $\mathcal{M} = (\mathcal{W}, \mathcal{R}_*, \mathcal{E}_*, \Vdash_p)$ be a pseudo-Fitting model for $\mathbf{JL}^{\mathbf{D}}$ and L be the set of labels used in derivations. An interpretation $[\cdot]$ based on model \mathcal{M} is a total function $[\cdot] : L \rightarrow \mathcal{W}$. Then we will define when an interpretation $[\cdot]$ validates a formula:

- $[\cdot]$ validates the labelled formula $w \Vdash A$, if $(\mathcal{M}, [w]) \Vdash_p A$,
- $[\cdot]$ validates the relational atom $wR_i v$, if $[w] \mathcal{R}_i [v]$.

Now consider a model \mathcal{M} for $\mathbf{JL}^{\mathbf{D}}$. A sequent $\Gamma \Rightarrow \Delta$ is valid for an interpretation $[\cdot]$ based on model \mathcal{M} , if whenever $[\cdot]$ validates all the formulas in Γ then it validates at least one formula in Δ . A sequent is valid in a model \mathcal{M} if it is valid for every interpretation based on \mathcal{M} . A sequent is valid in $\mathbf{JL}^{\mathbf{D}}$ (or it is $\mathbf{JL}^{\mathbf{D}}$ -valid), if it is valid in every $\mathbf{JL}^{\mathbf{D}}$ -model \mathcal{M} .

The following lemma is helpful in the rest of the paper.

Lemma 5. ([1]) The $\mathbf{JL}^{\mathbf{D}}$ -formula A is valid in model \mathcal{M} if and only if the sequent $\Rightarrow w \Vdash A$ is valid.

To have the fully explanatory property for the models of $\mathbf{JT}_n^{\mathbf{D}}$, we use the *total constant specification*:

$\mathcal{TCS} = \{ \llbracket c_{j_m}^{i_m} \rrbracket_{i_m} \dots \llbracket c_{j_1}^{i_1} \rrbracket_{i_1} A \mid A \text{ is an axiom of } \mathbf{JT}_n^{\mathbf{D}}, c_{j_i}^{i_i} \text{ are justification constants and } i_1, \dots, i_m \text{ are in } \mathbf{G} \}$.

Clearly, \mathcal{TCS} is axiomatically appropriate. In the following, let $\square \in \{\mathcal{K}_1, \dots, \mathcal{K}_n, \mathcal{D}\}$.

Theorem 2 (Realization Theorem). $\mathbf{JT}_n^{\mathbf{D}^\circ} = \mathbf{T}_n^{\mathbf{D}}$.

Proof. The proof of one direction, i.e. $\mathbf{JT}_n^{\mathbf{D}^\circ} \subseteq \mathbf{T}_n^{\mathbf{D}}$, is given by Lemma 4. Now, suppose $\mathbf{T}_n^{\mathbf{D}} \vdash F$. Then, there is a cut-free derivation Π of the sequent $\Rightarrow w \Vdash F$ in $\mathbf{G3T}_n^{\mathbf{D}}$, for some label w . Let $L = \{w_1, \dots, w_m\}$ ($m \geq 1$) be the set of all labels used in the derivation Π . By the method described below, we can find a formula F^r (the realization of F) in the language of $\mathbf{JT}_n^{\mathbf{D}}$ and a sequent $\Rightarrow w \Vdash F^r$, that is valid in the finite canonical model \mathcal{M}_{F^r} respecting \mathcal{TCS} , and therefore $\mathbf{JT}_n^{\mathbf{D}}(\mathcal{TCS}) \vdash F^r$.

Step 1. Construct disjoint families of related \square 's in the derivation Π (occurrences of \square in the related formulas in the premise and conclusion of rules are related). Negative and positive occurrences of \square in a sequent $\Gamma \Rightarrow \Delta$ are defined similar to the negative and positive parts of $\bigwedge \Gamma \rightarrow \bigvee \Delta$. Note that in a cut-free derivation, all occurrences of \square in a family have the same polarity, thus we can define negative and positive family of \square 's. A family is called *essential* if it contains at least one instance of \square introduced by the rule $(R\square)$, i.e., by the rules (RK_i) and (RD) (the \square introduced by $(R\square)$ is also called essential).

Step 2. Replace all occurrences of \square in each non-essential family by distinct justification variables, as follows (for simplicity, suppose A has no occurrences of \square):

$$\mathcal{K}_i A \rightsquigarrow \llbracket x^i \rrbracket_i A$$

$$\mathcal{D} A \rightsquigarrow \llbracket y^{\mathcal{D}} \rrbracket_{\mathcal{D}} A$$

where $x^i \in Tm_i$ and $y^{\mathcal{D}} \in Tm_{\mathcal{D}}$ are fresh justification variables. Suppose we have an infinite set of provisional variables x_i^* and $x_{i,j}^*$, for $i, j = 1, 2, \dots$. Each variable x_i^* is devoted to an essential \square . Now, replace all occurrences of \square in an essential family f by the term $x_1^* +_* \dots +_* x_{n_f}^*$, as follows:

$$\mathcal{K}_i A \rightsquigarrow \llbracket x_1^i +_i \dots +_i x_{n_f}^i \rrbracket_i A$$

$$\mathcal{D} A \rightsquigarrow \llbracket x_1^{\mathcal{D}} +_{\mathcal{D}} \dots +_{\mathcal{D}} x_{n_f}^{\mathcal{D}} \rrbracket_{\mathcal{D}} A$$

where A has no occurrences of \square , and x_i^* 's are fresh provisional justification variables, and n_f is the number of the rules $(R\square)$ which introduce \square 's in the essential family.¹ The substitutions of this step is called a *pre-realization* of boxes. We denote the pre-realization of formula F by F^{pr} .

Step 3. Construct the finite canonical model $\mathcal{M}_{F^{pr}} = (\mathcal{W}_{F^{pr}}, \mathcal{R}_*, \mathcal{E}_*, \Vdash_p)$, as in the proof of Theorem 1, in which $\mathcal{W}_{F^{pr}} = \{\Sigma_1, \dots, \Sigma_n\}$, where $\Sigma_i = \text{Con}_{\mathbf{JT}_n^{\mathbf{D}}(\mathcal{TCS})}(S_i)$, for F^{pr} -maximal $\mathbf{JT}_n^{\mathbf{D}}(\mathcal{TCS})$ -consistent set S_i . It is obvious that there are finitely many interpretations $[\cdot] : L \rightarrow \mathcal{W}_{F^{pr}}$ based on $\mathcal{M}_{F^{pr}}$. Let $I = \{[\cdot]_1, \dots, [\cdot]_l\}$ be the set of all such interpretations. Now, replace each provisional variable x_i^* by the justification term p_i , where $p_i = x_{i,1}^* +_* x_{i,2}^* +_* \dots +_* x_{i,l}^*$ and $x_{i,1}^*, x_{i,2}^*, \dots, x_{i,l}^*$ are provisional variables. Notice that $p_i \in Tm_*$. Now we have a tree \mathcal{T} of sequents in the language of $\mathbf{JT}_n^{\mathbf{D}}$.

Step 4. Let m be the number of all instances of $(R\square)$ in Π . We have two cases:

(i) Suppose $m = 0$, i.e., Π contains no application of $(R\square)$. Pick an arbitrary interpretation $[\cdot]_j$ from I . Beginning from initial sequents, we inductively prove that each sequent in \mathcal{T} is valid for $[\cdot]_j$. Initial sequents are obviously valid for $[\cdot]_j$. For the induction step, we only check the rules (LK_i) and (Ref_i) . Consider an instance of the rule (LK_i) for $i \in G$. After pre-realizing the modalities \mathcal{K}_i by a justification variable in Tm_i , say x , the rule looks like:

$$\frac{v \Vdash A, w \Vdash \llbracket x \rrbracket_i A, wR_i v, \Gamma \Rightarrow \Delta}{w \Vdash \llbracket x \rrbracket_i A, wR_i v, \Gamma \Rightarrow \Delta}$$

¹ It is useful to have an enumeration of all essential \square 's in the derivation Π , such as $\square_1, \square_2, \dots, \square_e$, where e is the total number of all essential \square 's in Π , and then devote variables x_1^*, \dots, x_e^* , respectively, to them. In this case, occurrences of \square in an essential family f should be replaced by the term $x_{i_1}^* +_* \dots +_* x_{i_{n_f}}^*$.

Suppose we have $(\mathcal{M}_{F^{pr}}, [w]_j) \Vdash_p \llbracket x \rrbracket_i A$, $[w]_j \mathcal{R}_i [v]_j$ and $[\cdot]_j$ validates all formulas in Γ . We need to show that $[\cdot]_j$ validates at least one formula in Δ . By the induction hypothesis, the premise $v \Vdash A, w \Vdash \llbracket x \rrbracket_i A, wR_i v, \Gamma \Rightarrow \Delta$ is valid for $[\cdot]_j$. From $(\mathcal{M}_{F^{pr}}, [w]_j) \Vdash_p \llbracket x \rrbracket_i A$ and $[w]_j \mathcal{R}_i [v]_j$ we conclude that $(\mathcal{M}_{F^{pr}}, [v]_j) \Vdash_p A$. Thus, the antecedent of $v \Vdash A, w \Vdash \llbracket x \rrbracket_i A, wR_i v, \Gamma \Rightarrow \Delta$ is valid for $[\cdot]_j$, and hence, $[\cdot]_j$ validates at least one formula in Δ as desired.

Next consider an instance of (Ref_i) :

$$\frac{wR_i w, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} (Ref_i)$$

Suppose $[\cdot]_j$ validates all formulas in Γ . Since the accessibility relation \mathcal{R}_i in $\mathcal{M}_{F^{pr}}$ is reflexive, for every $w \in L$ we have $[w]_j \mathcal{R}_i [w]_j$. Thus, $[\cdot]_j$ validates all formulas in Γ and $wR_i w$. By the induction hypothesis, the premise $wR_i w, \Gamma \Rightarrow \Delta$ is valid for $[\cdot]_j$. Hence, $[\cdot]_j$ validates at least one formula in Δ as desired.

The induction step for other rules is similar. Since $[\cdot]_j$ is arbitrary, the endsequent $\Rightarrow w \Vdash F^{pr}$ is validated in every interpretation from I . Now, put the realization r to be the pre-realization constructed in Step 2, i.e., $F^r = F^{pr}$. Thus, $\Rightarrow w \Vdash F^r$ is valid in $\mathcal{M}_{F^{pr}}$, and therefore $\mathbf{JT}_n^D(\mathcal{TCS}) \vdash F^r$.

(ii) Suppose $m > 0$. Consider an instance of (RK_i) in Π :

$$\frac{\Pi' \quad wR_i v, \Gamma \Rightarrow \Delta, v \Vdash A}{\Gamma \Rightarrow \Delta, w \Vdash \mathcal{K}_i A} (RK_i)$$

which have number h in the numbering of all rules (RK_i) from a given essential family f , and has been pre-realized by the term $p_h = x_{h,1}^i +_i \cdots +_i x_{h,j}^i +_i \cdots +_i x_{h,l}^i$ in Step 3. Thus, the rule (RK_i) , after pre-realization, is of the form:

$$\frac{\Pi'' \quad wR_i v, \Gamma \Rightarrow \Delta, v \Vdash A}{\Gamma \Rightarrow \Delta, w \Vdash \llbracket p_1 +_i \cdots +_i p_h +_i \cdots +_i p_{n_f} \rrbracket_i A}$$

Similar to case (i), beginning from initial sequents, we inductively prove that each sequent in Π'' is valid for interpretations of I . Suppose further that all instances of $(R\Box)$ in Π'' have been already realized using all interpretations from I . Consider an interpretation $[\cdot]_j$ from I such that validates all formulas in Γ . By the induction hypothesis, the premise $wR_i v, \Gamma \Rightarrow \Delta, v \Vdash A$ is validated in $[\cdot]_j$. Let Σ_k be an arbitrary element of $\mathcal{W}_{F^{pr}}$ such that $[w]_j \mathcal{R}_i \Sigma_k$ and $[\cdot]_j'$ be the interpretation identical to $[\cdot]_j$ except possibly on v , where we put $[v]_j' = \Sigma_k$. Clearly $[\cdot]_j'$ validates all the formulas in the antecedent of the premise, so it validates a formula in Δ or it validates $v \Vdash A$, i.e., $(\mathcal{M}_{F^{pr}}, \Sigma_k) \Vdash_p A$. In the former case, since v is not in Δ , $[\cdot]_j$ validates a formula in Δ . In the latter, since Σ_k is arbitrary and by Proposition 1, $\mathcal{M}_{F^{pr}}$ is fully explanatory, there is a term $t_{h,j}$ in $\mathcal{T}m_i$ such that $(\mathcal{M}_{F^{pr}}, [w]_j) \Vdash_p \llbracket t_{h,j} \rrbracket_i A$. Replace all occurrences of the variable $x_{h,j}^i$ (in p_h) in the tree \mathcal{T} by the term $t_{h,j}$. On the other hand, we have

$$(\mathcal{M}_{F^{pr}}, [w]_j) \Vdash_p \llbracket x_{h,1}^i +_i \cdots +_i t_{h,j} +_i \cdots +_i x_{h,l}^i \rrbracket_i A$$

and hence

$$(\mathcal{M}_{F^{pr}}, [w]_j) \Vdash_p \llbracket p_1 +_i \cdots +_i (x_{h,1}^i +_i \cdots +_i t_{h,j} +_i \cdots +_i x_{h,l}^i) +_i \cdots +_i p_{n_f} \rrbracket_i A$$

For all interpretations $[\cdot]_1, \dots, [\cdot]_l$ in I , repeat the above stage l times to find terms $t_{h,1}, \dots, t_{h,l}$ for the essential \mathcal{K}_i . Eventually, each term p_h is replaced by the term $q_h = t_{h,1} +_* \cdots +_* t_{h,l}$.

In a similar way, by the fully explanatory property of $\mathcal{M}_{F^{pr}}$, we find terms q_1, \dots, q_{n_f} , for other essential \Box 's in the family f , and replace each term p_j , where $1 \leq j \leq n_f$, by the term q_j . Thus, each instance of (RK_i)

$$\frac{wR_i v, \Gamma \Rightarrow \Delta, v \Vdash A}{\Gamma \Rightarrow \Delta, w \Vdash \mathcal{K}_i A}$$

is transformed into

$$\frac{wR_iv, \Gamma^r \Rightarrow \Delta^r, v \Vdash A^r}{\Gamma^r \Rightarrow \Delta^r, w \Vdash \llbracket q_1 +_i \cdots +_i q_{n_f} \rrbracket_i A^r}$$

where X^r denotes the realization of X .

Now, consider an instance of (RD) in Π :

$$\frac{\Pi' \quad wR_1v, \dots, wR_nv, \Gamma \Rightarrow \Delta, v \Vdash A}{\Gamma \Rightarrow \Delta, w \Vdash \mathcal{D}A} \quad (RD)$$

which have number h in the numbering of all rules (RD) from a given essential family f , and has been pre-realized by the term $p_h = x_{h,1}^{\mathcal{D}} +_{\mathcal{D}} \cdots +_{\mathcal{D}} x_{h,j}^{\mathcal{D}} +_{\mathcal{D}} \cdots +_{\mathcal{D}} x_{h,l}^{\mathcal{D}}$ in Step 3. Thus, the rule (RD) , after pre-realization, is of the form:

$$\frac{\Pi'' \quad wR_1v, \dots, wR_nv, \Gamma \Rightarrow \Delta, v \Vdash A}{\Gamma \Rightarrow \Delta, w \Vdash \llbracket p_1 +_i \cdots +_i p_h +_i \cdots +_i p_{n_f} \rrbracket_{\mathcal{D}} A}$$

Suppose that all instances of $(R\Box)$ in Π'' have been already realized using all interpretations from I , and $[\cdot]_j$ is an arbitrary interpretation from I that validates all formulas in Γ . By the induction hypothesis, the premise $wR_1v, \dots, wR_nv, \Gamma \Rightarrow \Delta, v \Vdash A$ is validated in $[\cdot]_j$. Let Σ_k be an arbitrary element of \mathcal{W} such that $[w]_j \mathcal{R}_1 \Sigma_k, \dots, [w]_j \mathcal{R}_n \Sigma_k$ and $[\cdot]'$ be the interpretation identical to $[\cdot]_j$ except possibly on v , where we put $[v]' = \Sigma_k$. Clearly $[\cdot]'$ validates all the formulas in the antecedent of the premise, so it validates a formula in Δ or it validates $v \Vdash A$, i.e. $(\mathcal{M}_{F^{pr}}, \Sigma_k) \Vdash_p A$. In the former case, since v is not in Δ , $[\cdot]_j$ validates a formula in Δ . In the latter, since Σ_k is arbitrary and by assumption $\mathcal{M}_{F^{pr}}$ is fully explanatory, by clause 2 of Proposition 1, there is a term $t_{h,j}$ in $Tm_{\mathcal{D}}$ such that $(\mathcal{M}_{F^{pr}}, [w]_j) \Vdash_p \llbracket t_{h,j} \rrbracket_{\mathcal{D}} A$. Now we replace all the variables $x_{h,j}^{\mathcal{D}}$ in the tree by the term $t_{h,j}$. On the other hand, we have

$$(\mathcal{M}_{F^{pr}}, [w]_j) \Vdash_p \llbracket x_{h,1}^{\mathcal{D}} +_{\mathcal{D}} \cdots +_{\mathcal{D}} t_{h,j} +_{\mathcal{D}} \cdots +_{\mathcal{D}} x_{h,l}^{\mathcal{D}} \rrbracket_{\mathcal{D}} A$$

and hence

$$(\mathcal{M}_{F^{pr}}, [w]_j) \Vdash_p \llbracket p_1 +_{\mathcal{D}} \cdots +_{\mathcal{D}} (x_{h,1}^{\mathcal{D}} +_{\mathcal{D}} \cdots +_{\mathcal{D}} t_{h,j} +_{\mathcal{D}} \cdots +_{\mathcal{D}} x_{h,l}^{\mathcal{D}}) +_{\mathcal{D}} \cdots +_{\mathcal{D}} p_{n_f} \rrbracket_{\mathcal{D}} A$$

For all interpretations $[\cdot]_1, \dots, [\cdot]_l$ in I , repeat the above stage l times to find terms $t_{h,1}, \dots, t_{h,l}$ for the essential \mathcal{D} . Eventually, each term p_h is replaced by the term $q_h = t_{h,1} +_{\mathcal{D}} \cdots +_{\mathcal{D}} t_{h,l}$. In a similar way, by the fully explanatory property of $\mathcal{M}_{F^{pr}}$, we find terms q_1, \dots, q_{n_f} , for other essential \mathcal{D} 's in the family f , and replace each term p_j , where $1 \leq j \leq n_f$, by the term q_j .

Eventually, we obtain a normal realization r and a tree T^r of sequents, with the endsequent $\Rightarrow w \Vdash F^r$. It is easy to verify that $\Rightarrow w \Vdash F^r$ is valid in $\mathcal{M}_{F^{pr}}$. But in order to show $\mathbf{JT}_{\mathbf{n}}^{\mathbf{D}}(\mathcal{TCS}) \vdash F^r$, we need to prove that the sequent $\Rightarrow w \Vdash F^r$, or the formula F^r , is valid in \mathcal{M}_{F^r} . Let σ be the substitution function which maps each provisional variable x_h^* , from Step 2, into q_h , simultaneously. Hence $\sigma(A) = A[q_1/x_1^*] \dots [q_e/x_e^*]$, where e is the number of all essential \Box 's in Π .

Construct the finite canonical model $\mathcal{M}_{F^r} = (\mathcal{W}_{F^r}, \mathcal{R}_*, \mathcal{E}_*, \Vdash_p)$. Notice that $F^r = \sigma(F^{pr})$, and hence, by the definition of subformulas (1), $X_{F^r} = \sigma(X_{F^{pr}})$ (where for a set X , $\sigma(X) = \{\sigma(A) \mid A \in X\}$). Thus, the number of elements of X_{F^r} and $X_{F^{pr}}$ are equal, $|X_{F^r}| = |X_{F^{pr}}|$. It is not difficult to show that F^r -maximal $\mathbf{JT}_{\mathbf{n}}^{\mathbf{D}}(\mathcal{TCS})$ -consistent sets are of the form $\sigma(S)$, where S is an F^{pr} -maximal $\mathbf{JT}_{\mathbf{n}}^{\mathbf{D}}(\mathcal{TCS})$ -consistent set. Thus $|\mathcal{W}_{F^r}| = |\mathcal{W}_{F^{pr}}|$. Suppose $\mathcal{W}_{F^r} = \{\Pi_1, \dots, \Pi_n\}$, and $[\cdot] : L \rightarrow \mathcal{W}_{F^r}$ is an arbitrary interpretation based on \mathcal{M}_{F^r} . Then similar

to Step 4, one can inductively prove that each sequent of \mathcal{T}^r is valid for $[\cdot]$. We only check the induction step that corresponds to the realization of an instance of the rule ($R\mathcal{K}_i$):

$$\frac{wR_iv, \Gamma \Rightarrow \Delta, v \Vdash A}{\Gamma \Rightarrow \Delta, w \Vdash \llbracket q_1 +_i \cdots +_i q_h +_i \cdots +_i q_{n_f} \rrbracket_i A}$$

in which the essential \mathcal{K}_i has number h in the essential family f . Suppose $[w] = \Pi_m \in \mathcal{W}_{F^r}$, where $\Pi_m = \text{Con}_{\mathbf{JT}_n^D(\mathcal{TCS})}(T_m)$. Also, assume $[\cdot]$ validates all formulas in Γ . By the induction hypothesis, the premise $wR_iv, \Gamma \Rightarrow \Delta, v \Vdash A$ is validated in $[\cdot]$. Let Π_k be an arbitrary element of \mathcal{W}_{F^r} such that $\Pi_m \mathcal{R}_i \Pi_k$ and $[\cdot]'$ be the interpretation identical to $[\cdot]$ except possibly on v , where we put $[v]' = \Pi_k$. Clearly, $[\cdot]'$ validates all the formulas in the antecedent of the premise, so it validates a formula in Δ or it validates $v \Vdash A$, i.e. $(\mathcal{M}_{F^r}, \Pi_k) \Vdash_p A$. In the former case, since v is not in Δ , $[\cdot]$ validates a formula in Δ . In the latter, pick an interpretation $[\cdot]_j \in I$ such that $[w]_j = \Sigma_m = \text{Con}_{\mathbf{JT}_n^D(\mathcal{TCS})}(S_m)$ and $T_m = \sigma(S_m)$. By Step 4, we know that $(\mathcal{M}_{F^r}, \Sigma_m) \Vdash \llbracket t_{h,j} \rrbracket_i A$. Thus $\Sigma_m \in \mathcal{E}(t_{h,j}, A)$, and hence $\llbracket t_{h,j} \rrbracket_i A \in \Sigma_m$. Therefore $\mathbf{JT}_n^D(\mathcal{TCS}), S_m \vdash \llbracket t_{h,j} \rrbracket_i A$. Then, by the Substitution Lemma (using substitution σ), we have $\mathbf{JT}_n^D(\mathcal{TCS}), T_m \vdash \llbracket t_{h,j} \rrbracket_i A$ (note that $t_{h,j}$ and A contain no occurrences of provisional variables x_1^*, \dots, x_e^*). Thus $\llbracket t_{h,j} \rrbracket_i A \in \Pi_m$, and by the Truth Lemma, $(\mathcal{M}_{F^r}, \Pi_m) \Vdash_p \llbracket t_{h,j} \rrbracket_i A$ (note that $\llbracket t_{h,j} \rrbracket_i A \in \text{Sub}(F^r)$). Therefore $(\mathcal{M}_{F^r}, \Pi_m) \Vdash_p \llbracket q_h \rrbracket_i A$, and hence

$$(\mathcal{M}_{F^r}, \Pi_m) \Vdash_p \llbracket q_1 +_i \cdots +_i q_h +_i \cdots +_i q_{n_f} \rrbracket_i A.$$

Finally, by Lemma 5, the formula F^r is valid in \mathcal{M}_{F^r} . Therefore $\mathbf{JT}_n^D(\mathcal{TCS}) \vdash F^r$. \square

We leave the proof of the realization theorem for $\mathbf{JS4}_n^D$ and $\mathbf{JS5}_n^D$ for future work.

References

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